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Non-Differentiable Mechanical Model and Its Implications

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Abstract Considering that the motions of the particles take place on fractals, a non-differentiable mechanical model is built. Only if the spatial coordinates are fractal functions, the Galilean version of our model is obtained: the geodesics satisfy a Navier-Stokes-type of equation with an imaginary viscosity coefficient for a complex speed field or respectively, a Schrödinger-type of equation or hydrodynamic equations, in the case of irrotational movements. Moreover, in this approach, the analysis of the fractal fluid dynamics generates conductive properties in the case of movements synchronization both on differentiable and fractal scales, and convective properties in the absence of synchronization (*e.g.* laser ablation plasma is analyzed). On the other hand, if both the spatial and temporal coordinates are fractal functions, it results that, the geodesics satisfy a Klein-Gordon-type of equation on a Minkowskian manifold.

Keywords Fractal fluids · Complex speed field · Navier-Stokes-type equation · Schrödinger-type equation

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1 Introduction

The complex dynamical systems which display chaotic behaviour are recognized to acquire self-similarity and manifest strong fluctuations on all possible scales (for details see [1–8]). Since the fractality appears as a universal property of the complex systems, it is necessary to create a new physics (fractal physics), either using the non-differentiability (for mathematical procedures see [9, 10] and for physical implications see [3, 11, 12]) or the transfinite sets (see, for example, the Cantor set and its physical implications (El Naschie's model) [4, 5, 13–17]). Consequently both the scale relativity (SR) [3, 11, 12] and the transfinite physics [4, 5, 13–17] are in this way developed.

The SR is build by completing the standard laws of classical physics (motion in space-time) with new scale laws in which "the space-time resolutions are used as intrinsic variables, playing for scale transformations the same role as played by velocities for motion transformations" [3, 11, 12]. This model is based both on the fractal space-time concept and a generalization of Einstein's principle of relativity to scale transformations. The space-time resolutions are redefined to characterize the scale's state of the reference systems, in the same way as velocity characterizes their state of motion. Also it is a requirement that the motion is governed by the laws of physics that apply for any state of the reference system (principle of motion-relativity) and the same principle applies to the scale (principle of scale-relativity). Mathematically, the principle of SR is achieved using the principle of scale-covariance (*i.e.* the simplest form of the physics equations under transformations of resolution) [3, 11, 12].

Three scales of interaction of SR were developed: (i) A "Galilean" version corresponding to the standard fractals with constant fractal dimensions [1, 2] which involves quantum mechanics [3, 11, 12, 18–20]; (ii) a special relativistic-scale version which implies the high energy physics [3]; (iii) a "general relativistic-scale" version which implies the cosmology [3, 21–23].

In the present paper some implications in the multi-particle systems dynamics using an extended SR model (for movements on non-differentiable curves (fractal curves) with an arbitrary and constant fractal dimension) both in its Galilean and relativistic versions are analyzed. The present paper is organized as follows: in Sect. 2 the Galilean version of the non-differentiable mechanical model is build: Sect. 2.1 presents mathematical and physical aspects in fractal space-time; Sect. 2.2 introduces geodesics followed by hydrodynamic equations presented in Sect. 2.3; fractal laws of conservation, conductive- and convective (*e.g.* ablation plasma behaviour)-type behaviours of a fractal fluid are analyzed in Sect. 2.4. In Sect. 3 paragraph the relativistic version of the non-differentiable mechanical model is elaborated.

2 Galilean Version of the Non-Differentiable Mechanical Model

2.1 Mathematical and Physical Aspects in Fractal Space-Time

Let us assume that the motion of particles take place on continuous but non-differentiable curves (fractal curves) with an arbitrary and constant fractal dimension, D_F. The non-differentiability, according with Cresson's mathematical procedures [6, 7] and Nottale's physical principles [3, 11, 12], states the followings:

(i) a continuous and a non-differentiable curve (or almost nowhere differentiable) is explicitly scale dependent, and its length tends to infinity, when the scale interval tends to zero.

In other words, a continuous and non-differentiable space is a fractal, in the general meaning given by Mandelbrot to this concept [1];

(ii) there is an infinity of fractals curves (geodesics) linking any couple of its points (or starting from any point) and being valid for all scales;

(iii) the breaking of local differential time reflection invariance. The time-derivative of a coordinate field $\boldsymbol{X}$ can be written two-fold:

$$\frac{d\boldsymbol{X}}{dt} = \lim_{dt\to 0} \frac{\boldsymbol{X}(t+dt) - \boldsymbol{X}(t)}{dt} = \lim_{dt\to 0} \frac{\boldsymbol{X}(t) - \boldsymbol{X}(t-dt)}{dt} \tag{1}$$

Both definitions are equivalent in the differentiable case. In the non-differentiable situation these definitions fail because the limits are no longer defined. "In the framework of scale relativity, the physics is related to the behaviour of the function during the "zoom" operation on the time resolution δt, here identified with the differential element dt ("substitution principle"), which is considered as an independent variable. The standard coordinate field $\boldsymbol{X}(t)$ is therefore replaced by the fractal coordinate field $\boldsymbol{X}(t, dt)$ (for details see [6, 7]) explicitly dependent on the time resolution interval, whose derivative is undefined only at the unobservable limit $dt \to 0$" [3]. The quantity t is a parameter along the geodesics and not a differentiable measure. As a consequence this leads us to define the two derivatives of the fractal coordinate field as explicit functions of the two variables t and dt:

$$\begin{aligned} \frac{d_+\boldsymbol{X}}{dt} &= \lim_{dt\to 0_+} \frac{\boldsymbol{X}(t+dt, dt) - \boldsymbol{X}(t, dt)}{dt} \\ \frac{d_-\boldsymbol{X}}{dt} &= \lim_{dt\to 0_-} \frac{\boldsymbol{X}(t, dt) - \boldsymbol{X}(t-dt, dt)}{dt} \end{aligned} \tag{2}$$

The sign, $+$, corresponds to the forward process and, $-$, to the backward process;

(iv) the differential of the fractal coordinate field $d\boldsymbol{X}(t, dt)$ can be expressed as the sum of two differentials: one of which is not scale-dependent (classical part $d_\pm \boldsymbol{x}(t)$), the other depends on it (fractal part $d_\pm \boldsymbol{\xi}(t, dt)$), therefore [6, 7]

$$d_\pm \boldsymbol{X}(t, dt) = d_\pm \boldsymbol{x}(t) + d_\pm \boldsymbol{\xi}(t, dt) \tag{3}$$

(v) the fractal part of $d\boldsymbol{X}$, satisfies the relation [6, 7]

$$d_\pm \xi_i \approx dt^{\frac{1}{D_F}} \tag{4}$$

or the equality:

$$\left(\frac{d_\pm \xi_i}{\lambda}\right) = \left(\frac{dt}{\tau}\right)^{\frac{1}{D_F}} \tag{5}$$

Similar expressions are given in [24, 25].

Written as

$$d_\pm \xi_i = \frac{\lambda}{\tau}\left(\frac{dt}{\tau}\right)^{(\frac{1}{D_F})-1} dt \tag{6}$$

the equations (5) involve the temporal scales δt and τ, also the length scale, λ, respectively. The significances of the time dt and τ result from the Random Walk (Brownian motion) or its generalization, Levy motion [3]. The differential time dt is identified with the resolution time (“substitution principle” [3, 11, 12]), $\delta t \equiv dt$, while τ corresponds to the fractal—non-fractal transition time. The measure λ is a characteristic length, for example of Planck’s type or de Broglie’s type (for details see [3]);

(vi) the local differential time reflection invariance is obtained by combining the two derivatives, d_+/dt and d_-/dt, in the complex operator [3, 11, 12]

$$\frac{\hat{d}}{dt} = \frac{1}{2}\left(\frac{d_+ + d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+ - d_-}{dt}\right) \tag{7}$$

We call this procedure “an extension by differentiability” (Cresson’s extension—for detail see [6, 7, 18]).

Applying this operator to the “position vector” generates a complex speed

$$\begin{aligned}
\boldsymbol{V} = \frac{\hat{d}\boldsymbol{X}}{dt} &= \frac{1}{2}\left(\frac{d_+\boldsymbol{X} + d_-\boldsymbol{X}}{dt}\right) - \frac{i}{2}\left(\frac{d_+\boldsymbol{X} - d_-\boldsymbol{X}}{dt}\right) \\
&= \frac{1}{2}\left(\frac{d_+\boldsymbol{x} + d_-\boldsymbol{x}}{dt} + \frac{d_+\boldsymbol{\xi} + d_-\boldsymbol{\xi}}{dt}\right) - \frac{i}{2}\left(\frac{d_+\boldsymbol{x} - d_-\boldsymbol{x}}{dt} + \frac{d_+\boldsymbol{\xi} - d_-\boldsymbol{\xi}}{dt}\right) \\
&= \boldsymbol{v} - i\boldsymbol{u}
\end{aligned} \tag{8}$$

with

$$\begin{aligned}
\boldsymbol{v} &= \frac{1}{2}\left(\frac{d_+\boldsymbol{x} + d_+\boldsymbol{\xi}}{dt} + \frac{d_-\boldsymbol{x} + d_\boldsymbol{\xi}}{dt}\right) = \frac{1}{2}(\boldsymbol{v}_+ + \boldsymbol{v}_-) \\
\boldsymbol{u} &= \frac{1}{2}\left(\frac{d_+\boldsymbol{x} + d_+\boldsymbol{\xi}}{dt} - \frac{d_-\boldsymbol{x} + d_-\boldsymbol{\xi}}{dt}\right) = \frac{1}{2}(\boldsymbol{v}_+ - \boldsymbol{v}_-) \\
\boldsymbol{v}_+ &= \frac{1}{2}\left(\frac{d_+\boldsymbol{x} + d_+\boldsymbol{\xi}}{dt}\right) \qquad \boldsymbol{v}_- = \frac{1}{2}\left(\frac{d_-\boldsymbol{x} + d\boldsymbol{\xi}}{dt}\right)
\end{aligned} \tag{9}$$

The real part, $\boldsymbol{v}$, of the complex speed $\boldsymbol{V}$, represents the standard classical speed, which is both differentiable and independent of resolution, while the imaginary part, $\boldsymbol{u}$, is a new quantity arising from fractality both non-differentiable and resolution-dependent. For the physical implications of a complex speed-field see references [26, 27];

(vii) “in order to account for the infinity of geodesics in the bundle, for their fractality and for the two values of the derivative which all come from the non-differentiable geometry of the space-time continuum, one therefore adopts a generalized statistical fluid like description where, instead of a classical deterministic speed or of a classical fluid speed field, one uses a doublet of fractal functions of spaces coordinates and time which are also explicit functions of resolution time” [3, 11, 12]. Consequently, the average values of the quantities must be considered in the previously mentioned sense [3, 11, 12]. Particularly, the average of $d_\pm\boldsymbol{X}$ is

$$\langle d_\pm\boldsymbol{X}\rangle = d_\pm\boldsymbol{x} \tag{10}$$

with

$$\langle d_\pm\boldsymbol{\xi}\rangle = 0 \tag{11}$$

(viii) using such an interpretation, the “particles”, are identified with the geodesics themselves. As a consequence, any measurement is interpreted as a sorting out process (or selection) of the geodesics by the measuring device [3].

Let us now assume that the movement curves (continuous but non-differentiable) are immersed in a 3-dimensional space, and that $\boldsymbol{X}$ of components X^i ($i = \overline{1,3}$) is the position vector of a point on the curve. Let us also consider a function $f(\boldsymbol{X}, t)$ and the following Taylor series expansion, up to the second order:

$$d_\pm f = \frac{\partial f}{\partial t} dt + \nabla f \cdot d_\pm \boldsymbol{X} + \frac{1}{2} \frac{\partial^2 f}{\partial X^i \partial X^j} d_\pm X^i d_\pm X^j \tag{12}$$

The expressions (12) are valid in any point of the space-time manifold and also for the points "$\boldsymbol{X}$" on the fractal curve that we have selected in relations (12).

From here, the forward and backward average values of this relation, take the form:

$$\langle d_\pm f \rangle = \left\langle \frac{\partial f}{\partial t} dt \right\rangle + \langle \nabla f \cdot d_\pm \boldsymbol{X} \rangle + \frac{1}{2} \left\langle \frac{\partial^2 f}{\partial X^i \partial X^j} d_\pm X^i d_\pm X^j \right\rangle \tag{13}$$

We stipulate that the mean values of the function f and its derivates coincide with themselves and the differentials $d_\pm X^i$ and dt are independent. Therefore the averages of their products coincide with the product of averages. Consequently, (13) become:

$$d_\pm f = \frac{\partial f}{\partial t} dt + \nabla f \langle d_\pm \boldsymbol{X} \rangle + \frac{1}{2} \frac{\partial^2 f}{\partial X^i \partial X^j} \langle d_\pm X^i d_\pm X^j \rangle \tag{14}$$

or more, using (3) with the properties (11),

$$d_\pm f = \frac{\partial f}{\partial t} dt + \nabla f d_\pm \boldsymbol{x} + \frac{1}{2} \frac{\partial^2 f}{\partial X^i \partial X^j} \left(d_\pm x^i d_\pm x^j + \langle d\xi^i_\pm d\xi^j_\pm \rangle \right) \tag{15}$$

Even if the average value of the fractal coordinate, $d\xi^i_\pm$, is null (see (11)), for the higher order of the fractal coordinate average, the situation can be different. Let us focus on the mean expression $\langle d\xi^i_\pm d\xi^j_\pm \rangle$. If $i \neq j$ the value of the average is zero due the independence of $d\xi^i$ and $d\xi^j$. So, using (6) we can write (see also [18, 19]):

$$\langle d\xi^i_\pm d\xi^j_\pm \rangle = \pm \delta^{ij} \frac{\lambda^2}{\tau} \left(\frac{dt}{\tau} \right)^{(\frac{2}{D_F})-1} dt \tag{16}$$

with

$$\delta^{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where we had considered that:

$$\langle d\xi^i_+ d\xi^j_+ \rangle > 0 \quad \text{and} \quad dt > 0$$
$$\langle d\xi^i_- d\xi^j_- \rangle < 0 \quad \text{and} \quad dt < 0$$

Then, (15) may be rewritten under the form:

$$d_\pm f = \frac{\partial f}{\partial t} dt + \nabla f d_\pm \boldsymbol{x} + \frac{1}{2} \frac{\partial^2 f}{\partial X^i \partial X^j} d_\pm x^i d_\pm x^j \pm \frac{\partial^2 f}{\partial X^i \partial X^j} \delta^{ij} \frac{\lambda^2}{2\tau} \left(\frac{dt}{\tau} \right)^{(\frac{2}{D_F})-1} dt \tag{17}$$

If we divide by dt, and neglect the terms which contain differential factors (for details on the method see [18, 19]), (17) are simplified to:

$$\frac{d_{\pm} f}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{v}_{\pm} \nabla f \pm \frac{\lambda^2}{2\tau} \left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta f \tag{18}$$

with $\nabla^2 = \sum_i \frac{\partial^2}{\partial X_i^2}$. These expressions also allow us to define the operator,

$$\frac{d_{\pm}}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_{\pm} \nabla \pm \frac{\lambda^2}{2\tau} \left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta \tag{19}$$

—see also [18, 19]. Under these circumstances, let us calculate $\hat{d} f/dt$. Taking into account (7), (8), and (19), we obtain:

$$\begin{aligned}
\frac{\hat{d} f}{dt} &= \frac{1}{2}\left[\left(\frac{d_+ f}{dt} + \frac{d_- f}{dt}\right) - i\left(\frac{d_+ f}{dt} - \frac{d_- f}{dt}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_+ \nabla f + \frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta f\right)\right. \\
&\quad \left. + \left(\frac{\partial f}{\partial t} + \boldsymbol{v}_- \nabla f - \frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta f\right)\right] \\
&\quad - \frac{i}{2}\left[\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_+ \nabla f + \frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta f\right)\right. \\
&\quad \left. - \left(\frac{\partial f}{\partial t} + \boldsymbol{v}_- \nabla f - \frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta f\right)\right] \\
&= \frac{\partial f}{\partial t} + \left(\frac{\boldsymbol{v}_+ + \boldsymbol{v}_-}{2} - i\,\frac{\boldsymbol{v}_+ - \boldsymbol{v}_-}{2}\right)\nabla f - i\,\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta f \\
&= \frac{\partial f}{\partial t} + \boldsymbol{V} \cdot \nabla f - i\,\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta f
\end{aligned} \tag{20}$$

This equation also allows us to define the fractal operator:

$$\frac{\hat{d}}{dt} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla - i\,\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta \tag{21}$$

We now apply the principle of scale covariance (for details see [3, 11, 12]), and postulate that the conversion from classical (differentiable) mechanics to the "fractal" mechanics considered here can be implemented by replacing the standard time-derivative, d/dt, by the complex operator $(\hat{d}/dt)$ (this result is the principle of scale covariance given by Nottale in [3, 11, 12]). The operator $(\hat{d}/dt)$ plays the role of a 'covariant derivative operator' and it is used to write the fundamental equation of dynamics under the same form as in the classical and differentiable case.

2.2 Geodesics

The inertial principle in its covariant form (Nottale's principle [3, 11, 12]) is reduced to a Navier-Stokes type of equation (geodesics equation),

$$\frac{\hat{d}\boldsymbol{V}}{dt} = \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} - i\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta \boldsymbol{V} = 0 \tag{22}$$

with an imaginary viscosity coefficient ν

$$\nu = i\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \tag{23}$$

This means that the local complex acceleration field, $\partial \boldsymbol{V}/\partial t$, the convective term, $\boldsymbol{V} \cdot \nabla \boldsymbol{V}$ and the dissipative one, $\Delta \boldsymbol{V}$, reciprocally compensate at any point on the fractal curve. Moreover, the behaviour of the fractal fluid is visco-elastic or hysteretic type. Such results are in agreement with [28, 29]: the fractal fluid can be described by Kelvin-Voight or Maxwell rheological model with imaginary structure coefficient ν.

In the case of the irrotational motions, *i.e.*:

$$\nabla \times \boldsymbol{V} = 0 \tag{24}$$

the speed field (8) can be expressed through the gradient of a scalar function Φ,

$$\boldsymbol{V} = \nabla \Phi \tag{25}$$

The parameter Φ represents the scalar potential of the complex speed field, $\Phi = \mathrm{Re}\,\Phi + i\,\mathrm{Im}\,\Phi$.

Substituting (25) in (22) it results

$$\nabla\left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - i\frac{\lambda^2}{\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta\Phi\right] = 0 \tag{26}$$

and by integration, a Bernoulli type equation

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - i\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \Delta\Phi = F(t) \tag{27}$$

with $F(t)$ a function that is only time dependent. Particularly, for Φ given by:

$$\Phi = -i\frac{\lambda^2}{\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \ln\psi \tag{28}$$

where ψ is a new complex scalar function, (27) takes the form

$$\frac{\lambda^4}{4\tau^2}\left(\frac{dt}{\tau}\right)^{(\frac{4}{D_F})-2} \Delta\psi + i\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \frac{\partial\psi}{\partial t} + \frac{F(t)}{2}\psi = 0 \tag{29}$$

From here, "Schrödinger" type geodesics result for $F(t) \equiv 0$, *i.e.*

$$\frac{\lambda^4}{4\tau^2}\left(\frac{dt}{\tau}\right)^{(\frac{4}{D_F})-2} \Delta\psi + i\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \frac{\partial\psi}{\partial t} = 0 \tag{30}$$

Particularly, for movements on Peano-type fractal curves, *i.e.* in the fractal dimension $D_F = 2$, using Compton's length and temporal scales:

$$\lambda = \frac{\hbar}{2m_0 c} \qquad \tau = \frac{\hbar}{m_0 c^2} \tag{31}$$

equation (30) takes the Schrödinger standard form

$$\frac{\hbar^2}{2m_0}\Delta\psi + i\hbar\frac{\partial\psi}{\partial t} = 0 \tag{32}$$

2.3 Hydrodynamic model

By replacing the complex speed field (8) in (22), and by separating the real and imaginary parts, we obtain

$$m_0\frac{\partial \boldsymbol{v}}{\partial t} + m_0\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla(Q) \tag{33a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla(\boldsymbol{v}\cdot\boldsymbol{u}) + \frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\Delta\boldsymbol{v} = 0 \tag{33b}$$

where Q is the fractal potential,

$$Q = -\frac{m_0\boldsymbol{u}^2}{2} - m_0\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\nabla\cdot\boldsymbol{u} \tag{34}$$

The explicit form of the complex speed field is given by the expression

$$\psi = \sqrt{\rho}e^{iS} \tag{35}$$

with ρ the amplitude and S the phase. Then (28) with

$$\Phi = -i\frac{\lambda^2}{\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\ln(\sqrt{\rho}e^{iS}) \tag{36}$$

involves the complex velocity field components

$$\boldsymbol{v} = \frac{\lambda^2}{\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\nabla S \qquad \boldsymbol{u} = \frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\nabla\ln\rho \tag{37}$$

while the fractal potential (34) is given by the simple expression

$$Q = -m_0\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{38}$$

—for other details see [30, 31].

By using (37), (33b) takes the form:

$$\nabla\left(\frac{\partial\ln\rho}{\partial t} + \boldsymbol{v}\cdot\nabla\ln\rho + \nabla\cdot\boldsymbol{v}\right) = 0 \tag{39}$$

or by integration with $\rho \neq 0$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = T(t) \tag{40}$$

with $T(t)$ a function which is only time dependent.

Equation (33a) corresponds to the momentum conservation law, while (40), with $T(t) \equiv 0$, to the probability density conservation law. Consequently the equations:

$$\begin{aligned} &m_0\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\right) = -\nabla(Q) \\ &\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \end{aligned} \tag{41}$$

with Q given by (34), form the fractal hydrodynamic equations in the fractal dimension D_F. The fractal potential (34) is induced by the non-differentiable space-time (for more details see [22]). Our results are more general comparing to those obtained by Nottale (the movements take place on a Peano's type fractal curve at Compton's space-time scale) [3, 11, 12].

In an external potential U, the fractal hydrodynamic equations become

$$m_0\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\right) = -\nabla(Q + U) \tag{42a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{42b}$$

Two types of states are distinguished:

(i) Dynamic states. For $\partial/\partial t = 0$ and $\boldsymbol{v} \neq 0$, *i.e.* at the differentiable scale, (42a, 42b) give

$$\nabla\left(\frac{m_0 \boldsymbol{v}^2}{2} - \frac{m_0 \boldsymbol{u}^2}{2} - m_0 \frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \nabla \cdot \boldsymbol{u} + U\right) = 0 \tag{43a}$$

$$\nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{43b}$$

specifically,

$$\frac{m_0 \boldsymbol{v}^2}{2} - \frac{m_0 \boldsymbol{u}^2}{2} - m_0 \frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \nabla \cdot \boldsymbol{u} + U = E \tag{44a}$$

$$\rho \boldsymbol{v} = \nabla \times \boldsymbol{F} \tag{44b}$$

Consequently, the non-fractal inertia, $m_0 \boldsymbol{v} \cdot \nabla \boldsymbol{v}$, the external force, $-\nabla U$, and the fractal force, $-\nabla Q$, are balanced in every field point—equation (43a). The sum of the non-fractal kinetic energy, $m\boldsymbol{v}^2/2$, external potential, U, and fractal potential, Q, is invariant, *i.e.*, equal to the integration constant $E \neq E(\boldsymbol{r})$—equation (44a). The parameter $E \equiv \langle E \rangle$ represents the total energy of the dynamic system. The probability flow density $\rho \boldsymbol{V}$ has no sources—equation (43b), *i.e.*, its streamlines are closed—equation (44b).

(ii) Static states. For $\partial/\partial t = 0$ and $\boldsymbol{v} = 0$, *i.e.* at non-differentiable scale equations (42a, 42b) give

$$\nabla\left(-\frac{m_0 \boldsymbol{u}^2}{2} - m_0 \frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1} \nabla \cdot \boldsymbol{u} + U\right) = 0 \tag{45a}$$

i.e.

$$-\frac{m_0\boldsymbol{u}^2}{2}-m_0\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\nabla\cdot\boldsymbol{u}+U=E \tag{45b}$$

Consequently, the fractal force, $-\nabla Q$, and external force, $-\nabla U$, are balanced in every field point—equation (45a). The sum of the external potential, U, and fractal potential, Q, is invariant, *i.e.*, equal to the integration constant $E\neq E(\boldsymbol{r})$—equation (45b). The parameter $E\equiv\langle E\rangle$ represents the total energy of the static system. Equation (42b) is identically satisfied.

2.4 Conservation Laws. Conductive- and Convective-type Behaviours

Let us apply the complex operator $\hat{\partial}/\partial t$—equation (21) to an arbitrary fractal function (for details see [5, 18, 19]), $\varepsilon=\varepsilon(\boldsymbol{r},t)$. It results

$$\frac{\hat{d}\varepsilon}{dt}=\frac{\partial\varepsilon}{\partial t}+\boldsymbol{V}\nabla\varepsilon-i\frac{\lambda^2}{2\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\Delta\varepsilon=0 \tag{46}$$

or more, by separating the real and the imaginary parts,

$$\frac{\partial\varepsilon}{\partial t}+\boldsymbol{v}\cdot\nabla\varepsilon=0 \tag{47a}$$

$$-\boldsymbol{u}\cdot\nabla\varepsilon=\frac{\lambda^2}{\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\Delta\varepsilon \tag{47b}$$

Consequently, at the differentiable scale, the local temporal variation, $\partial\varepsilon/\partial t$, and the term $\boldsymbol{v}\cdot\nabla\varepsilon$, are equal, while at the non-differentiable scale, the terms, $\boldsymbol{u}\cdot\nabla\varepsilon$ and $\Delta\varepsilon$ compensate each other. Particularly, for $\boldsymbol{v}=\boldsymbol{u}$ ("synchronic" movements at different scales), from (61) we obtain the diffusion type equation

$$\frac{\partial\varepsilon}{\partial t}=\frac{\lambda^2}{\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\Delta\varepsilon \tag{48}$$

Such equation is a direct result of the Fourier type law

$$\boldsymbol{j}(\varepsilon)=\frac{\lambda^2}{\tau}\left(\frac{dt}{\tau}\right)^{(\frac{2}{D_F})-1}\nabla\varepsilon \tag{49}$$

where $\boldsymbol{j}(\varepsilon)$ is the current density. Therefore, (48) and (49) describe a fractal mechanism of conduction type at different scales.

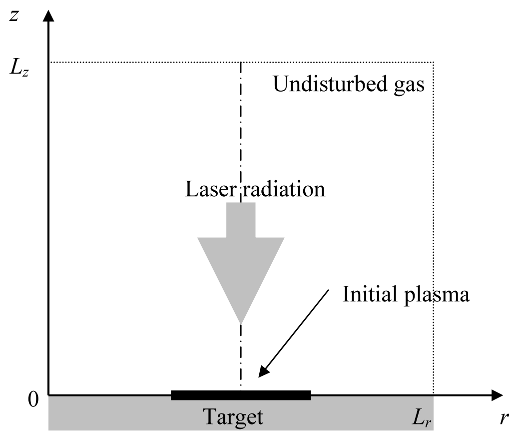
Moreover, by multiplying (42b) with ε, *i.e.*

$$\frac{\partial(\rho\varepsilon)}{\partial t}+\nabla\cdot(\rho\varepsilon\boldsymbol{v})=\rho\left(\frac{\partial\varepsilon}{\partial t}+\boldsymbol{v}\cdot\nabla\varepsilon\right), \tag{50}$$

and taking into account (47a), the conservation law for ε is found in the form

$$\frac{\partial(\rho\varepsilon)}{\partial t}+\nabla\cdot(\rho\varepsilon\boldsymbol{v})=0 \tag{51}$$

Fig. 1 Integration domain used for the numerical simulation of the laser-produced aluminium plasma expansion



Particularly, if ε is the energy density of a fluid [32], $\varepsilon = e + (p/\rho) + v^2/2$ (where e is the internal energy and p the pressure of the fluid), the "classical" form of the energy conservation law results.

Let us now apply the previous considerations in the numerical simulations of plasma expansion produced by the laser ablation. The plasma expansion induced in the region above the target surface (Fig. 1) has axial symmetry and is analyzed in a cylindrical coordinates system. The z-axis coincides with the laser beam axis and is directed along the outer normal to the target surface. The plasma evolution is described having made the following assumptions: (i) the plasma is in a local thermo-dynamical equilibrium and satisfies the quasi-neutrality condition; (ii) the expansion is described in the approximation of a non-viscous non-thermo-conducting gas; (iii) the energy loss by thermal radiation is neglected and the ideal gas equations of state are considered; (iv) the source term is introduced through the boundary conditions.

In such circumstances, the two-dimensional gas dynamics is described by (42a, 42b) with $\nabla Q = -\nabla p/\rho$ and the energy balance equation (51), *i.e.* explicitly

$$\begin{aligned}
&\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rnu) + \frac{\partial}{\partial z}(nv) = 0\\
&\frac{\partial (nu)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rnu^2) + \frac{\partial}{\partial z}(nuv) = -\frac{\partial p}{\partial r}\\
&\frac{\partial (nv)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rnuv) + \frac{\partial}{\partial z}(nv^2) = -\frac{\partial p}{\partial z}\\
&\frac{\partial (ne)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rnue) + \frac{\partial}{\partial z}(nve) = -p\left[\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial v}{\partial z}\right]
\end{aligned} \tag{52}$$

Here, t is the time, r and z the spatial coordinates, n the atoms density, u and v the velocity vector components.

For the numerical integration, the following initial and boundary conditions are considered:

(i) The box integration domain is initially filled with undisturbed gas,

$$\begin{aligned}
t = 0: \quad & u = v = 0 \quad n = n_0\\
& T = T_0 \quad 0 \le (r \times z) \le (L_r \times L_z)
\end{aligned} \tag{53}$$

where T is the temperature;

(ii) The interaction of the laser beam with the target produces a plasma source located on the target surface—assumed to have a Gaussian space-time profile,

$$\begin{aligned} z=0: \quad & u=v=0 \quad T=T_{plasma} \\ & n=n_{\max}\exp\left[-\frac{(t-\tau)^2}{(\tau_L/2)^2}\right]\exp\left[-\frac{(r-L_r/2)^2}{(d_L/2)^2}\right] \end{aligned} \tag{54}$$

with d_L, τ_L similarly with the laser beam space-time full widths, and $T_{plasma}=11.8$ eV the initial plasma temperature. We underline that the ablation takes place only into a region with the characteristic diameter of about 100 µm The maximum atoms density $n_{\max}$ is taken with respect to the critical electron density ($n_{ec}=3.9\times10^{21}$ cm^{-3} [33]) at the laser wavelength ($\lambda=532$ nm) and the average ions charge state $\bar{Z}=2$ (for details see [34]);

(iii) The symmetry conditions are

$$\begin{aligned} L_r: \quad & u(0)=u(L_r) \quad v(0)=v(L_r) \\ & n(0)=n(L_r) \quad T(0)=T(L_r) \quad r=0 \end{aligned} \tag{55}$$

and the undisturbed gas is considered to be on the upper boundary

$$z=L_z: \quad u=v=0 \quad n=n_0 \quad T=T_0 \tag{56}$$

The equations system (52) together with the conditions (53–56) is numerically solved using finite differences [35] and the following parameters:

$$\begin{aligned} & L_r=L_z=300\ \mu\text{m} \quad \tau_L=10\ \text{ns} \quad d_L=100\ \mu\text{m} \quad n_{\max}=1.95\times10^{21}\ \text{cm}^{-3} \\ & n_0=n_{\max}/1000 \quad T_0=0.1\ \text{eV} \end{aligned}$$

In Figs. 2(a–d) the 2D-contour curves of the total atom density at specified time moments, exactly as obtained from the numerical simulation, are given. The convective-type behaviour of the plasma plum due to the non-synchronization of motions at both scales (differentiable and non-differentiable) results in its "mushroom"-type expansion. On the contrary, at a different time scale, the synchronization of motions at both differentiable and non-differentiable scales implies a conductive-type behaviour of the plasma structure (for details see (48) for $\varepsilon=n$). In this case, plasma will disappear by a diffusion-type mechanism.

It results that the shape of the plasma core is in agreement with the experimental observations from [36]. Moreover, the "mushroom" type structure from Figs. 2(a–e) and the vortical motion arising at the plume periphery are also in good agreement with the experimental observations from [37, 38]. Therefore, the numerical simulations describe quantitatively well the convective behaviour of the plasma at the early stages of its evolution (at nanosecond time scale).

Similar numerical results but at a different time scale are given in [36]. Both in the presented approach and in [36] the estimated profile is imposed by the Gaussian space-time profile of the laser beam. As a result, our numerical simulations will emphasize this behaviour.

3 Relativistic Version of the Non-differentiable Mechanical Model

Most elements of the approach summarized in Sect. 2 remain correct in the relativistic motion case, with the time, t, replaced by the proper time, σ, as a curvilinear parameter along

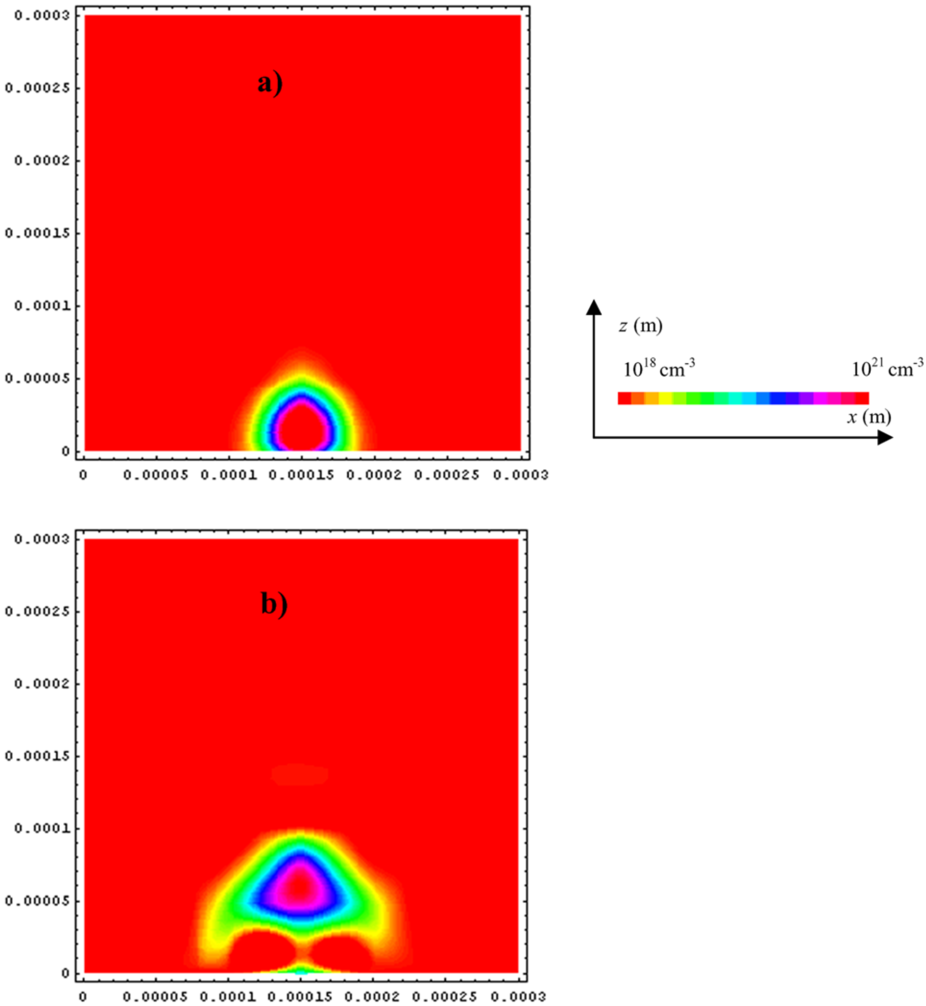


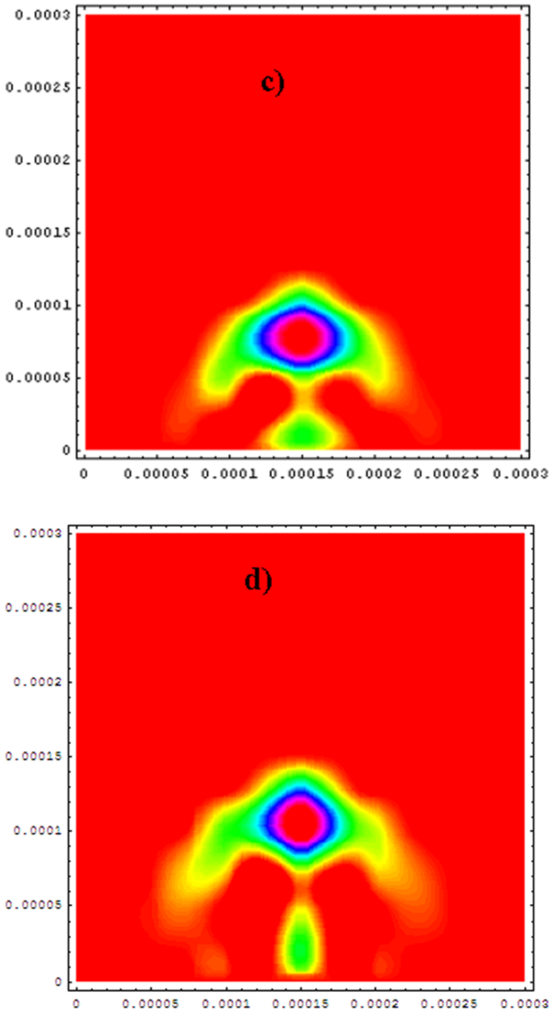
Fig. 2 The numerical solutions for the total atom density (n_T) in laser produced plasma expansion at the time moments (**a**) $t = 4$ ns, (**b**) $t = 8$ ns, (**c**) $t = 12$ ns; (**d**) $t = 16$ ns. As a result of the non-synchronization of motions at both differentiable and non-differentiable scales, the plasma plum presents itself as an expanding "mushroom"

the geodesics. Now, not only the spatial coordinate, X^i, $i = 1, 2, 3$, but also the "temporal" coordinate, X^0, is considered to be non-differentiable and consequently, a fractal. Therefore, we shall consider that the points on the motion curve are given by the fractal coordinates field X^μ, $\mu = \overline{0, 3}$.

The non-differentiability implies the followings:

(i) the breaking of the reflection invariance at the infinitesimal level. The standard four-coordinates field $X^\mu(\sigma)$ is therefore replaced by the fractal four-coordinates field $X^\mu(\sigma, d\sigma)$. As a consequence, this leads us to define the two derivatives of the fractal four-

Fig. 2 (*Continued*)



coordinates field as explicit functions of the two variables σ and $d\sigma$ (scale resolution):

$$\begin{aligned}\frac{d_+X^\mu}{d\sigma} &= \lim_{d\sigma\to 0_+}\frac{X^\mu(\sigma+d\sigma,d\sigma)-X^\mu(\sigma,d\sigma)}{d\sigma}\\ \frac{d_-X^\mu}{d\sigma} &= \lim_{d\sigma\to 0_-}\frac{X^\mu(\sigma,d\sigma)-X^\mu(\sigma-d\sigma,d\sigma)}{d\sigma}\end{aligned} \tag{57}$$

The sign "+" corresponds to the forward process and "−", to the backward process;

(ii) the differential of the fractal four-coordinates field $dX^\mu(\sigma,d\sigma)$ can be expressed as the sum of two differentials, one which is not scale-dependent (classical part $d_\pm x^\mu(\sigma)$) and the other dependent on it (fractal part $d_\pm\xi^\mu(\sigma,d\sigma)$), therefore:

$$d_\pm X^\mu(\sigma,d\sigma) = d_\pm x^\mu(\sigma) + d_\pm\xi^\mu(\sigma,d\sigma) \tag{58}$$

(iii) the fractal part of dX^μ satisfies the expression:

$$d_\pm \xi^\mu \approx (d\sigma)^{\frac{1}{D_F}} \tag{59}$$

or more, the equality:

$$d_\pm \xi^\mu = \lambda_\pm^\mu (d\sigma)^{\frac{1}{D_F}} \tag{60}$$

with $\lambda_\pm^\mu$ constant coefficients;

(iv) the reflection invariance at the infinitesimal level is recovered by combining the two derivatives $d_+/d\sigma$ and $d_-/d\sigma$ within the complex operator:

$$\frac{\hat{d}}{d\sigma} = \frac{1}{2}\left(\frac{d_+ + d_-}{d\sigma}\right) - \frac{i}{2}\left(\frac{d_+ - d_-}{d\sigma}\right) \tag{61}$$

By applying this operator to the fractal four-coordinate field, a complex four-speed field is generated:

$$\begin{aligned} V^\mu = \frac{\hat{d}X^\mu}{d\sigma} &= \frac{1}{2}\left(\frac{d_+X^\mu + d_-X^\mu}{d\sigma}\right) - \frac{i}{2}\left(\frac{d_+X^\mu - d_-X^\mu}{d\sigma}\right) \\ &= \frac{1}{2}\left(\frac{d_+x^\mu + d_+\xi^\mu}{d\sigma} + \frac{d_-x^\mu + d_-\xi^\mu}{d\sigma}\right) - \frac{i}{2}\left(\frac{d_+x^\mu + d_+\xi^\mu}{d\sigma} - \frac{d_-x^\mu + d_-\xi^\mu}{d\sigma}\right) \\ &= \frac{1}{2}(v_+^\mu + v_-^\mu) - \frac{i}{2}(v_+^\mu - v_-^\mu) = v^\mu - iu^\mu \end{aligned} \tag{62}$$

The real part, v^μ, represents the standard classical four speed which is differentiable and independent of resolution, while the imaginary part, u^μ, is a new quantity arising from fractality both non-differentiable and resolution-dependent;

(v) the average of $d_\pm X^\mu$ is:

$$\langle d_\pm X^\mu \rangle = d_\pm x^\mu \tag{63}$$

with

$$\langle d_\pm \xi^\mu \rangle = 0 \tag{64}$$

Let us assume that the movement curves are immersed in a four-dimensional space-time, and that X^μ is the "position" four-vector of a point on the curve. Let us also consider a function $f(X^\mu, \sigma)$ and the following Taylor expansion up to the second order:

$$d_\pm f = \frac{\partial f}{\partial \sigma} d\sigma + \frac{\partial f}{\partial X^\mu} d_\pm X^\mu + \frac{1}{2}\frac{\partial^2 f}{\partial X^\mu \partial X^\nu} d_\pm X^\mu d_\pm X^\nu \tag{65}$$

The relations (65) are valid in any point of the space-time manifold and also for the point "X^μ" on the fractal curve previously selected by us in the expressions (65).

From here, the forward and the backward average values of this relation, according with the method given in Sect. 2, take the successive forms:

$$\langle d_\pm f \rangle = \left\langle \frac{\partial f}{\partial \sigma} d\sigma \right\rangle + \left\langle \frac{\partial f}{\partial X^\mu} d_\pm X^\mu \right\rangle + \frac{1}{2}\left\langle \frac{\partial^2 f}{\partial X^\mu \partial X^\nu} d_\pm X^\mu d_\pm X^\nu \right\rangle, \tag{66}$$

$$d_\pm f = \frac{\partial f}{\partial \sigma} d\sigma + \frac{\partial f}{\partial X^\mu} \langle d_\pm X^\mu \rangle + \frac{1}{2}\frac{\partial^2 f}{\partial X^\mu \partial X^\nu} \langle d_\pm X^\mu d_\pm X^\nu \rangle \tag{67}$$

or more, using (58) with the properties (64)

$$d_\pm f = \frac{\partial f}{\partial \sigma} d\sigma + \frac{\partial f}{\partial X^\mu} d_\pm x^\mu + \frac{1}{2} \frac{\partial^2 f}{\partial X^\mu \partial X^\nu} (d_\pm x^\mu d_\pm x^\nu + \langle d_\pm \xi^\mu d_\pm \xi^\nu \rangle) \tag{68}$$

Even if the average value of the fractal four-coordinates field, $d_\pm \xi^\mu$, is null (see (64)) for the higher order of the fractal four-coordinates field average, the situation can be different. Let us focus on the mean expression $\langle d_\pm \xi^\mu d_\pm \xi^\nu \rangle$. Due to the independence of $d\xi^\mu$ and $d\xi^\nu$, we can write:

$$\langle d_\pm \xi^\mu d_\pm \xi^\nu \rangle = \pm \lambda_\pm^\mu \lambda_\pm^\nu (d\sigma)^{\frac{2}{D_F}} \tag{69}$$

where we have considered that:

$$\langle d_+ \xi^\mu d_+ \xi^\nu \rangle > 0 \quad \text{and} \quad d\sigma > 0$$
$$\langle d_- \xi^\mu d_- \xi^\nu \rangle < 0 \quad \text{and} \quad d\sigma < 0$$

Then (68) may be written under the form:

$$d_\pm f = \frac{\partial f}{\partial \sigma} d\sigma + \frac{\partial f}{\partial X^\mu} d_\pm x^\mu + \frac{1}{2} \frac{\partial^2 f}{\partial X^\mu \partial X^\nu} d_\pm x^\mu d_\pm x^\nu \pm \frac{1}{2} \frac{\partial^2 f}{\partial X^\mu \partial X^\nu} \lambda_\pm^\mu \lambda_\pm^\nu (d\sigma)^{\frac{2}{D_F}} \tag{70}$$

If we divide by $d\sigma$ and neglect the terms which contain differential factors, (70) are simplified to:

$$\frac{d_\pm f}{d\sigma} = \frac{\partial f}{\partial \sigma} + v_\pm^\mu \frac{\partial f}{\partial X^\mu} \pm \frac{1}{2} \frac{\partial^2 f}{\partial X^\mu \partial X^\nu} \lambda_\pm^\mu \lambda_\pm^\nu (d\sigma)^{(\frac{2}{D_F})-1} \tag{71}$$

The correspondence with the physical reality allows us to choose:

$$\lambda_\pm^\mu \lambda_\pm^\nu = 2\delta^{\mu\nu} \lambda^2 \tag{72}$$

in which case (71) are reduced to:

$$\frac{d_\pm f}{d\sigma} = \frac{\partial f}{\partial \sigma} + v_\pm^\mu \frac{\partial f}{\partial X^\mu} \pm \lambda^2 (d\sigma)^{(\frac{2}{D_F})-1} \frac{\partial^2 f}{\partial (X^\mu)^2} \tag{73}$$

These expressions also allow us to define the operator:

$$\frac{d_\pm}{d\sigma} = \frac{\partial}{\partial \sigma} + v_\pm^\mu \frac{\partial}{\partial X^\mu} \pm \lambda^2 (d\sigma)^{(\frac{2}{D_F})-1} \frac{\partial^2}{\partial (X^\mu)^2} \tag{74}$$

Under these circumstances, let us calculate $\hat{d} f/d\sigma$. Taking into account (61), (62) and (74), we obtain:

$$\begin{aligned} \frac{\hat{d} f}{d\sigma} &= \frac{1}{2}\left[\left(\frac{d_+ f}{d\sigma} + \frac{d_- f}{d\sigma}\right) - i\left(\frac{d_+ f}{d\sigma} + \frac{d_- f}{d\sigma}\right)\right] \\ &= \frac{1}{2}\left[\left(\frac{\partial f}{\partial \sigma} + v_+^\mu \frac{\partial f}{\partial X^\mu} + \lambda^2 (d\sigma)^{(\frac{2}{D_F})-1} \frac{\partial^2 f}{\partial (X^\mu)^2}\right)\right. \\ &\quad \left. + \left(\frac{\partial f}{\partial \sigma} + v_-^\mu \frac{\partial f}{\partial X^\mu} - \lambda^2 (d\sigma)^{(\frac{2}{D_F})-1} \frac{\partial^2 f}{\partial (X^\mu)^2}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&-\frac{i}{2}\left[\left(\frac{\partial f}{\partial\sigma}+v_+^{\mu}\frac{\partial f}{\partial X^{\mu}}+\lambda^2(d\sigma)^{(\frac{2}{D_F})-1}\frac{\partial^2 f}{\partial(X^{\mu})^2}\right)\right.\\
&\left.-\left(\frac{\partial f}{\partial\sigma}+v_-^{\mu}\frac{\partial f}{\partial X^{\mu}}-\lambda^2(d\sigma)^{(\frac{2}{D_F})-1}\frac{\partial^2 f}{\partial(X^{\mu})^2}\right)\right]\\
&=\frac{\partial f}{\partial\sigma}+\left(\frac{v_+^{\mu}+v_-^{\mu}}{2}-i\frac{v_+^{\mu}-v_-^{\mu}}{2}\right)\frac{\partial f}{\partial X^{\mu}}-i\lambda^2(d\sigma)^{(\frac{2}{D_F})-1}\frac{\partial^2 f}{\partial(X^{\mu})^2}\\
&=\frac{\partial f}{\partial\sigma}+V^{\mu}\frac{\partial f}{\partial X^{\mu}}-i\lambda^2(d\sigma)^{(\frac{2}{D_F})-1}\frac{\partial^2 f}{\partial(X^{\mu})^2}
\end{aligned}
\tag{75}
$$

This relation allows us to define the fractal operator:

$$
\frac{\hat{d}}{d\sigma}=\frac{\partial}{\partial\sigma}+V^{\mu}\frac{\partial}{\partial X^{\mu}}-i\lambda^2(d\sigma)^{(\frac{2}{D_F})-1}\frac{\partial^2}{\partial(X^{\mu})^2}
\tag{76}
$$

To write the equation of motion we use a generalized equivalence principle (a generalization of Nottale's covariance principle). We obtain a geodesic equation in terms of the covariant derivative:

$$
\frac{\hat{d}V_{\nu}}{d\sigma}=\frac{\partial V_{\nu}}{\partial\sigma}+V^{\mu}\frac{\partial V_{\nu}}{\partial X^{\mu}}-i\lambda^2(d\sigma)^{(\frac{2}{D_F})-1}\frac{\partial^2 V_{\nu}}{\partial(X^{\mu})^2}=0
\tag{77}
$$

If V_{ν} does not explicitly depend on the parameter σ and has the form $V_{\nu}\approx i\partial(\ln\psi)/\partial X^{\nu}$, we can show that, on a Minkowskian manifold, using the method given in [12], the geodesics satisfy a Klein-Gordon-type equation.

4 Conclusions

By considering that the motion of the particles takes place on continuous but non-differentiable curves, *i.e.* on fractals, an extended SR model in a constant arbitrary fractal dimension D_F with second order terms in the equation of motion for the complex speed field in both its Galilean and relativistic versions, is obtained. The Galilean version resulted by considering that only the spatial coordinates are fractal functions. In this approach, the geodesics are described by a Navier-Stokes-type equation with an imaginary viscosity coefficient and, from here, in the particular case of irrotational movement, by a Schrödinger-type equation. The standard Schrödinger equation is obtained as a particular case of irrotational movement at Compton scale, in the constant fractal dimension $D_F=2$.

The extended fractal hydrodynamic model is obtained from a Navier-Stokes-type equation by separating the real and the imaginary parts of the complex speed field. Using a momentum transport equation and a probability density conservation law the model is generated.

The conservation laws, particularly the energy conservation law, are deduced for the fractal quantities. In such a context, the synchronization of the movements both at differentiable and non-differentiable scales involves a conductive-type behaviour for the fractal fluid. On the contrary, a non-synchronization of the movements generates a convective-type behaviour of the fractal fluid. The convective-type behaviour was analyzed by means of the numerical simulations of the plasma expansion produced by laser ablation using the hydrodynamic and energy equations. These numerical simulations describe quantitatively well the plasma behaviour at the early stages (at nanoseconds time scale).

By considering that both the spatial and temporal coordinates are fractal functions the relativistic version of the non-differentiable mechanical model was obtained. It resulted that, on a Minkowskian manifold, the geodesics satisfy a Klein-Gordon-type equation.

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